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The time evolution and the quantum fluctuation for two forced quantum oscillators with mixing of two modes

Jiang-Fan Li[1,2], **Fu-Liang Xiao**[1], **Zong-Fu Jiang**[2] **and Jia-Yuan Fang**[1]

[1] Department of Physics and Electronic Science, Changsha University of Science and Technology, Changsha, Hunan, 410077, People's Republic of China

[2] Institute of Science, National University of Defense Technology, Changsha, Hunan, 410073, People's Republic of China

E-mail: lijiang6211@yahoo.com.cn and lijf@csust.edu.cn

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Abstract

By using the Lewis–Riesenfeld quantum invariant theory and properly choosing Hermitian invariant operator, a closed solution of the Schrödinger equation is derived for two forced quantum oscillators with mixing of two modes, and the quantum fluctuations in the output fields are evaluated. For the initial two-mode squeezed number or squeezed coherent state, in some particular conditions, the time evolution of the oscillators can not only preserve the initial two-mode squeezing, but also produce squeezing in the individual modes; and exhibit a periodical squeezing behaviour. For the initial two-mode number state or coherent state, there is no squeezing in the individual and mutual quadrature phases of the two-mode fields. Furthermore, regardless of which state above being initially considered, the quantum fluctuations of all the quadrature phases in the output fields are all independent of the driving parameters. In particular, for the initial two-mode coherent state, the variances of the output fields are also independent of other parameters in the Hamiltonian, and always preserve their initial values $1/4$.

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1. Introduction

The model of two coupled time-dependent harmonic oscillators has received much attention due to its applications in quantum mechanics and quantum optics. For instance, it has been used to study photon statistics, squeezing, entanglement and the exchange of nonclassical properties between two modes of the electromagnetic field in optical parametric processes [1–16]. The model has been solved in various cases by adopting the Heisenberg equations of motion [1–9], Lie-group method [12, 13], the entangled state representation [14] etc. However, in the literature above, most of the systems or models were investigated only in those cases: namely, either the two-mode coupled term or the driving term of the Hamiltonians is zero, and the parameters of the Hamiltonians are partially time-dependent. Recently, by virtue of the Lewis–Riesenfeld invariant theory, a model has been developed [18] to describe a generalized non-degenerate optical parametric down conversion, whose Hamiltonian contains the above two terms which are arbitrarily time-dependent. Indeed, the Lewis–Riesenfeld invariant theory has proved powerful in analyzing the quantum mechanical behaviours and been applied to various time-dependent problems of quantum mechanics and quantum optics [18–28]. In this work, we shall solve two forced quantum oscillators with mixing of two modes by the theory of invariants, and investigate the quantum fluctuation of the output fields.

2. The dynamical system

The Hamiltonian of the time-dependent system in this study is (in natural units $\hbar = c = 1$)

$$\hat{H} = \hat{H}_0 + \hat{H}_1 + \hat{H}_2 \tag{2.1a}$$

$$\hat{H}_0 = \sum_{j=1}^{2} \omega_j(t)\hat{a}_j^+\hat{a}_j + G_0(t) \tag{2.1b}$$

$$\hat{H}_1 = \sum_{j=1}^{2} G_j(t)\left[\hat{a}_j^+ \exp(\mathrm{i}\varphi_j(t)) + \hat{a}_j \exp(-\mathrm{i}\varphi_j(t))\right] \tag{2.1c}$$

$$\hat{H}_2 = Q_{12}(t)\left[\hat{a}_1^+\hat{a}_2 \exp(\mathrm{i}\varphi_{12}(t)) + \hat{a}_1\hat{a}_2^+ \exp(-\mathrm{i}\varphi_{12}(t))\right]. \tag{2.1d}$$

Here $\hat{a}_j$ and $\hat{a}_j^+$ are the annihilation and creation operations for the mode j ($j = 1, 2$, j is assigned the same values in what follows), respectively, and satisfy the following commutation relations

$$\left[\hat{a}_j, \hat{a}_k^+\right] = \delta_{jk}, \qquad [\hat{a}_j, \hat{a}_k] = \left[\hat{a}_j^+, \hat{a}_k^+\right] = 0 \quad (j, k = 1, 2), \tag{2.2}$$

where $\omega_j(t)$, $Q_{12}(t)$, $\varphi_{12}(t)$, $G_0(t)$, $G_j(t)$ and $\varphi_j(t)$ are arbitrary real functions of time; $\hat{H}_0$ is the free Hamiltonian for the two-mode field; $\hat{H}_1$ is referred to as the driving term, $G_j(t)\exp(\mathrm{i}\varphi_j(t))$ can be regarded as a classical generalized driving force acting on the mode j [7]; $\hat{H}_2$ describes the mixing of two modes [8], $Q_{12}(t)$ and $\varphi_{12}(t)$ are arbitrary pump coupling parameters. When $G_j = 0$, $Q_{12}(t) = \lambda$ and $\varphi_{12}(t) = \nu t - \phi$, the Hamiltonian (2.1) reduces to the model in [15] or [6] ($\nu = \omega_L$, $\phi = 0$), which corresponds to the up-conversion process. When $G_j = 0$, $Q_{12}(t) = \lambda\gamma(t)$ and $\varphi_{12}(t) = 0$, Hamiltonian (2.1) reduces to the model of the article [16], which can be used to study the exchange of nonclassical properties. When $G_j \neq 0$ and $Q_{12}(t) = 0$, the Hamiltonian (2.1) corresponds to the forced quantum oscillators subject to transient classical driving force [29]. When all parameters are arbitrarily time-dependent functions, the Hamiltonian (2.1) may describe two forced quantum oscillators coupled by some interactions which cause two-mode mixing in quantum optics. Hereinafter, we will try to solve the Schrödinger equation for this general situation.

3. Solving the Schrödinger equation via time-independent invariant

The time evolution of the quantum states is governed by the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \tag{3.1}$$

According to the Lewis–Riesenfeld invariant theory [17], an operator $\hat{I}(t)$ that obeys the following invariant equation

$$\frac{\partial}{\partial t}\hat{I}(t) - \mathrm{i}[\hat{I}(t), \hat{H}(t)] = 0 \tag{3.2}$$

is called an invariant whose eigenvalue is time-independent. In this study, we construct the Hermitian invariant by using unitary transformation of Hermitian operator $\hat{K}_0 = \sum_{j=1}^{2} C_j \hat{a}_j^+ \hat{a}_j$, i.e.

$$\hat{I}(t) = \hat{D}_1(z_1(t))\hat{D}_2(z_2(t))\hat{V}(\varsigma(t)) \cdot \left(C_1\hat{a}_1^+\hat{a}_1 + C_2\hat{a}_2^+\hat{a}_2\right)\hat{V}^+(\varsigma(t))\hat{D}_2^+(z_2(t))\hat{D}_1^+(z_1(t)), \tag{3.3}$$

where C_1 and C_2 are arbitrary real constants. When $C_1 = C_2 = 1$, $\langle\hat{I}(t)\rangle$ corresponds to the sum of the photon numbers in the two modes, which maintains conservation in the evolution of a quantum system; and then the Hamiltonian (2.1) can describe a generalized frequency conversion process; when $C_1 = -C_2 = 1$ and $G_j(t) = 0$, the Hamiltonian (2.1) is equivalent to the model of article [16], which may be used to study the transference of a kind of nonclassical properties between two interacting modes of light. And besides, in (3.3), $\hat{D}_j(z_j(t))$ is the displacement operator for mode j defined by

$$\hat{D}_j(z_j(t)) = \exp\left[z_j(t))\hat{a}_j^+ - z_j^*(t))\hat{a}_j\right] \tag{3.4a}$$

$\hat{V}(\varsigma(t))$ is the two-mixed operator given by

$$\hat{V}(\varsigma) = \exp\left[\varsigma^*(t)\hat{a}_1^+\hat{a}_2 - \varsigma(t)\hat{a}_1\hat{a}_2^+\right], \tag{3.4b}$$

where

$$z_j(t) = r_j(t)\exp(\mathrm{i}\delta_j(t)) \tag{3.5a}$$

$$\varsigma(t) = r_{12}(t)\exp(\mathrm{i}\delta_{12}(t)). \tag{3.5b}$$

Parameters $r_j(t), \delta_j(t)$, $r_{12}(t)$ and $\delta_{12}(t)$ are the real functions of time. For brevity, $z_j(t)$, $r_{12}(t)$ and $\delta_{12}(t)$ are rewritten as z_j, r_{12} and δ_{12} (when $t = 0$, denoted as z_{j0}, r_{120}, and δ_{120}), and may be determined by (2.1) and (3.2). By using the following relational expressions (omitting their Hermitian conjugate formats) [16],

$$\hat{D}_j(z_j)\hat{a}_j\hat{D}_j^+(z_j) = \hat{a}_j - z_j \tag{3.6a}$$

$$\hat{V}(\varsigma)\hat{a}_1\hat{V}^+(\varsigma) = \hat{a}_1\cos r_{12} - \hat{a}_2\exp(-\mathrm{i}\delta_{12})\sin r_{12} \tag{3.6b}$$

$$\hat{V}(\varsigma)\hat{a}_2\hat{V}^+(\varsigma) = \hat{a}_2\cos r_{12} + \hat{a}_1\exp(\mathrm{i}\delta_{12})\sin r_{12}, \tag{3.6c}$$

(3.3) becomes

$$\hat{I}(t) = \sum_{j=1}^{2}\left(g_{\omega_j}\hat{a}_j^+\hat{a}_j + g_j\hat{a}_j + g_j^*\hat{a}_j^+\right) + q_{12}\hat{a}_1\hat{a}_2^+ + q_{12}^*\hat{a}_1^+\hat{a}_2 + g_0. \tag{3.7}$$

From (2.1) and (3.7), we obtain the following commutation relation

$$\begin{aligned}[\hat{I}(t), \hat{H}(t)] = \Bigg\{ &\sum_{j=1}^{2}\left\{g_j G_j(t)\exp(\mathrm{i}\varphi_j(t)) + \left[g_{\omega_j}G_j(t)\exp(\mathrm{i}\varphi_j(t))\right.\right.\\ &\left.\left. - g_j^*\omega_j(t)\right]\hat{a}_j^+\right\} + [q_{12}^*G_2(t)\exp(\mathrm{i}\varphi_2(t)) - g_2^*Q_{12}(t)\exp(\mathrm{i}\varphi_{12}(t))]\hat{a}_1^+\\ &+ [q_{12}G_1(t)\exp(\mathrm{i}\varphi_1(t)) - g_1^*Q_{12}(t)\exp(-\mathrm{i}\varphi_{12}(t))]\hat{a}_2^+\\ &+ \left[\left(g_{\omega_1} - g_{\omega_2}\right)Q_{12}(t)\exp(\mathrm{i}\varphi_{12}(t)) + q_{12}^*(\omega_2(t) - \omega_1(t))\right]\hat{a}_1^+\hat{a}_2\\ &+ q_{12}^*Q_{12}(t)\exp(-\mathrm{i}\varphi_{12}(t))\left(\hat{a}_1^+\hat{a}_1 - \hat{a}_2^+\hat{a}_2\right)\Bigg\} - \text{h.c.}\end{aligned} \tag{3.8}$$

where h.c. stands for Hermitian conjugate, and

$$g_{\omega_1} = c_1 \cos^2 r_{12} + c_2 \sin^2 r_{12} \tag{3.9a}$$

$$g_{\omega_2} = c_2 \cos^2 r_{12} + c_1 \sin^2 r_{12} \tag{3.9b}$$

$$q_{12} = \tfrac{1}{2}(c_2 - c_1)\,\mathrm{e}^{\mathrm{i}\delta_{12}} \sin 2r_{12}, \tag{3.9c}$$

$$g_1 = -z_1^*(c_1 \cos^2 r_{12} + c_2 \sin^2 r_{12}) + \tfrac{1}{2}(c_1 - c_2) z_2^*\,\mathrm{e}^{\mathrm{i}\delta_{12}} \sin 2r_{12} \tag{3.10a}$$

$$g_2 = -z_2^*(c_2 \cos^2 r_{12} + c_1 \sin^2 r_{12}) + \tfrac{1}{2}(c_1 - c_2) z_1^*\,\mathrm{e}^{-\mathrm{i}\delta_{12}} \sin 2r_{12} \tag{3.10b}$$

$$g_0 = z_1 z_1^*(c_1 \cos^2 r_{12} + c_2 \sin^2 r_{12}) + z_2 z_2^*(c_2 \cos^2 r_{12} + c_1 \sin^2 r_{12}) \\ + \tfrac{1}{2}(c_2 - c_1) \sin 2r_{12}(z_1 z_2^*\,\mathrm{e}^{\mathrm{i}\delta_{12}} + z_1^* z_2\,\mathrm{e}^{-\mathrm{i}\delta_{12}}). \tag{3.10c}$$

Substituting (3.7) and (3.8) into (3.2), we obtain the following equations (omitting their conjugate equations):

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} g_{\omega_1} + Q_{12}(t)[q_{12}^* \exp(-\mathrm{i}\varphi_{12}(t)) - q_{12} \exp(\mathrm{i}\varphi_{12}(t))] = 0 \tag{3.11a}$$

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} g_{\omega_2} + Q_{12}(t)[q_{12} \exp(\mathrm{i}\varphi_{12}(t)) - q_{12}^* \exp(-\mathrm{i}\varphi_{12}(t))] = 0 \tag{3.11b}$$

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} q_{12} + q_{12}[\omega_1(t) - \omega_2(t)] + Q_{12}(t)\left(g_{\omega_2} - g_{\omega_1}\right) \exp(-\mathrm{i}\varphi_{12}(t)) = 0 \tag{3.11c}$$

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} g_0 + G_1(t)[g_1 \exp(\mathrm{i}\varphi_1(t)) - g_1^* \exp(-\mathrm{i}\varphi_1(t))] + G_2(t)[g_2 \exp(\mathrm{i}\varphi_2(t)) \\ - g_2^* \exp(-\mathrm{i}\varphi_2(t))] = 0 \tag{3.12a}$$

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} g_1 + g_1\omega_1(t) + Q_{12}(t) g_2 \exp(-\mathrm{i}\varphi_{12}(t)) - G_1(t) g_{\omega_1} \exp(-\mathrm{i}\varphi_1(t)) \\ - G_2(t) q_{12} \exp(-\mathrm{i}\varphi_2(t)) = 0 \tag{3.12b}$$

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} g_2 + g_2\omega_2(t) + Q_{12}(t) g_1 \exp(\mathrm{i}\varphi_{12}(t)) - G_2(t) g_{\omega_2} \exp(-\mathrm{i}\varphi_2(t)) \\ - G_1(t) q_{12}^* \exp(-\mathrm{i}\varphi_1(t)) = 0. \tag{3.12c}$$

By substituting (3.9) into (3.11), we obtain two independent equations:

$$\frac{\mathrm{d}r_{12}}{\mathrm{d}t} = Q_{12}(t) \sin[\delta_{12} + \varphi_{12}(t)] \tag{3.13a}$$

$$\frac{\mathrm{d}\delta_{12}}{\mathrm{d}t} = \omega_1(t) - \omega_2(t) + 2Q_{12}(t) \cot 2r_{12} \cos[\varphi_{12}(t) + \delta_{12}]. \tag{3.13b}$$

Now, we consider the case of the frequency converter: $Q_{12}(t) = Q_{12}$, $\omega_1(t) = \omega_1$ and $\omega_2(t) = \omega_2$, where Q_{12}, ω_1 and ω_2 are the time-independent pump coupling constant and frequencies of signal and idler fields, respectively; $\varphi_{12}(t) = \omega t$, here ω is the frequency of the pump field. For the case of perfect energy matching, we have $\omega = \omega_2 - \omega_1$. Substituting above parameters into (3.13), we can obtain a special solution of (3.13)

$$r_{12} = Q_{12} t, \qquad \delta_{12} = \pi/2 - \omega t = \pi/2 + \omega_1 t - \omega_2 t. \tag{3.14}$$

In general, for arbitrary time-dependent parameters, r_{12} and δ_{12} can be solved firstly from (3.13), and then z_j can be obtained by substituting them into (3.12). Moreover, r_{12} and δ_{12} are dependent on the pump coupling parameters Q_{12}(t) and φ_{12}(t), but independent of the driving parameters G_j(t) and φ_j(t). Substituting the solution r_{12}, δ_{12} and z_j from (3.13) and (3.12) into (3.7), we can obtain the time-dependent invariant $\hat{I}(t)$.

Let $|n_1, n_2\rangle$ be the eigenstate vector of the operator $\hat{K}_0$, i.e.,

$$\hat{K}_0|n_1, n_2\rangle = (C_1 n_1 + C_2 n_2)|n_1, n_2\rangle. \tag{3.15}$$

From (3.3) and (3.15), it is seen that the states $\hat{D}_1(z_1)\hat{D}_2(z_2)\hat{V}(\varsigma)|n_1, n_2\rangle$ are eigenstate vectors of the operator $\hat{I}(t)$, and proved to be complete by using expression $\sum_{n_1,n_2}|n_1, n_2\rangle\langle n_1, n_2| = 1$. According to the Lewis–Riesenfeld quantum-invariant theory [17], the general solution of the time-dependent Schrödinger equation (3.1) can be expressed as

$$|\psi(t)\rangle = \sum_{n_1,n_2} C_{n_1n_2}\exp\left(\mathrm{i}\alpha_{n_1n_2}\right)\hat{D}_1(z_1)\hat{D}_2(z_2)\hat{V}(\varsigma)|n_1, n_2\rangle, \tag{3.16}$$

where $\alpha_{n_1n_2}(t)$ is Lewis–Riesenfeld phase, and can be decomposed into geometric phase $\gamma_{n_1n_2}(t)$ and dynamical phase $\beta_{n_1n_2}(t)$, namely

$$\alpha_{n_1n_2}(t) = \gamma_{n_1n_2}(t) + \beta_{n_1n_2}(t), \tag{3.17a}$$

where

$$\gamma_{n_1n_2}(t) = \int_0^t \langle n_1, n_2|\hat{V}^+(\varsigma)\hat{D}_2^+(z_2)\hat{D}_1^+(z_1)\mathrm{i}\frac{\partial}{\partial t}[\hat{D}_1(z_1)\hat{D}_2(z_2)\hat{V}(\varsigma)]|n_1, n_2\rangle\,\mathrm{d}t \tag{3.17b}$$

$$\beta_{n_1n_2}(t) = -\int_0^t \langle n_1, n_2|\hat{V}^+(\varsigma)\hat{D}_2^+(z_2)\hat{D}_1^+(z_1)\hat{H}(t)\hat{D}_1(z_1)\hat{D}_2(z_2)\hat{V}(\varsigma)|n_1, n_2\rangle\,\mathrm{d}t. \tag{3.17c}$$

By adopting the following relations (omitting their Hermitian conjugate formats)

$$\hat{V}^+(\varsigma)\hat{a}_1\hat{V}(\varsigma) = \hat{a}_1\cos r_{12} + \hat{a}_2\exp(-\mathrm{i}\delta_{12})\sin r_{12} \tag{3.18a}$$

$$\hat{V}^+(\varsigma)\hat{a}_2\hat{V}(\varsigma) = \hat{a}_2\cos r_{12} - \hat{a}_1\exp(\mathrm{i}\delta_{12})\sin r_{12} \tag{3.18b}$$

$$\hat{D}_j^+(z_j)\hat{a}_j\hat{D}_j(z_j) = \hat{a}_j + z_j \tag{3.18c}$$

$$\hat{V}^+(\varsigma)i\frac{\partial}{\partial t}\hat{V}(\varsigma) = \frac{1}{2}\hat{a}_1\hat{a}_2^+(\dot{\delta}_{12}\sin 2r_{12} - i2\dot{r}_{12})\exp(\mathrm{i}\delta_{12}) + \mathrm{h.c} - \left(\hat{a}_1^+\hat{a}_1 - \hat{a}_2^+\hat{a}_2\right)\dot{\delta}_{12}\sin^2 r_{12}, \tag{3.19}$$

a rather lengthy calculation yields the following results:

$$\gamma_{n_1n_2} = \int_0^t \left[\frac{\mathrm{i}}{2}(\dot{z}_1 z_1^* + \dot{z}_2 z_2^* - z_1\dot{z}_1^* - z_2\dot{z}_2^*) - (n_1 - n_2)\dot{\delta}_{12}\sin^2 r_{12}\right]\mathrm{d}t \tag{3.20a}$$

$$\begin{aligned}\beta_{n_1n_2} = -\int_0^t \mathrm{d}t\{&[\omega_1(t)n_1 + \omega_2(t)n_2]\cos^2 r_{12} + [\omega_1(t)n_2 + \omega_2(t)n_1]\sin^2 r_{12}\\ &+ \omega_1(t)z_1z_1^* + \omega_2(t)z_2z_2^* + Q_{12}(t)[z_1^*z_2\exp(\mathrm{i}\varphi_{12}(t))\\ &+ z_1z_2^*\exp(-\mathrm{i}\varphi_{12}(t)) - (n_1 - n_2)\cos(\delta_{12} + \varphi_{12}(t))\sin 2r_{12}]\\ &+ G_1(t)[z_1^*\exp(\mathrm{i}\varphi_1(t)) + z_1\exp(-\mathrm{i}\varphi_1(t))] + G_2(t)[z_2^*\exp(\mathrm{i}\varphi_2(t))\\ &+ z_2\exp(-\mathrm{i}\varphi_2(t))] + G(t)\},\end{aligned} \tag{3.20b}$$

where $\dot{z}_j = \frac{\mathrm{d}z_j}{\mathrm{d}t}$, $\dot{r}_{12} = \frac{\mathrm{d}r_{12}}{\mathrm{d}t}$ and $\dot{\delta}_{12} = \frac{\mathrm{d}\delta_{12}}{\mathrm{d}t}$. Substituting (3.20) into (3.17a), we may write the Lewis–Riesenfeld phase as

$$\alpha_{n_1n_2} = -(\varepsilon_1 n_1 + \varepsilon_2 n_2) + \sigma, \tag{3.21}$$

where

$$\varepsilon_1 = \int_0^t \{\dot{\delta}_{12}\sin^2 r_{12} + \omega_1(t)\cos^2 r_{12} + \omega_2(t)\sin^2 r_{12} - Q_{12}(t)\cos[\delta_{12} + \varphi_{12}(t)]\sin 2r_{12}\}\,\mathrm{d}t \tag{3.22a}$$

$$\varepsilon_2 = \int_0^t \{-\dot{\delta}_{12}\sin^2 r_{12} + \omega_2(t)\cos^2 r_{12} + \omega_1(t)\sin^2 r_{12} + Q_{12}(t)\cos[\delta_{12} + \varphi_{12}(t)]\sin 2r_{12}\}\,\mathrm{d}t \tag{3.22b}$$

$$\begin{aligned}\sigma = \int_0^t \bigg\{&\frac{\mathrm{i}}{2}(\dot{z}_1 z_1^* + \dot{z}_2 z_2^* - z_1\dot{z}_1^* - z_2\dot{z}_2^*) - \omega_1(t)z_1 z_1^* - \omega_2(t)z_2 z_2^* \\ &- Q_{12}(t)[z_1 z_2^*\exp(-\mathrm{i}\varphi_{12}(t)) + z_1^* z_2\exp(\mathrm{i}\varphi_{12}(t))] - G_1(t)[z_1^*\exp(\mathrm{i}\varphi_1(t)) \\ &+ z_1\exp(-\mathrm{i}\varphi_1(t))] - G_2(t)[z_2^*\exp(\mathrm{i}\varphi_2(t)) + z_2\exp(-\mathrm{i}\varphi_2(t))] - G(t)\bigg\}\,\mathrm{d}t.\end{aligned} \tag{3.22c}$$

If $\omega_1(t) = \omega_1$ and $\omega_2(t) = \omega_2$ are constants, from (3.22*a*) and (3.22*b*), we have

$$\varepsilon_1 + \varepsilon_2 = (\omega_1 + \omega_2)t. \tag{3.23}$$

At $t = 0$, the initial state vector of the system is

$$|\psi(0)\rangle = \hat{D}_1(z_{10})\hat{D}_2(z_{20})\hat{V}(\varsigma_0)\sum_{n_1,n_2} C_{n_1 n_2}|n_1, n_2\rangle. \tag{3.24}$$

Obviously, for the initial two-mode squeezed number states, $|\psi(0)\rangle = \hat{S}(\xi_0)|n_1, n_2\rangle$, we have

$$z_{10} = z_{20} = \varsigma_0 = r_{120} = 0, \tag{3.25}$$

where $\hat{S}(\xi_0)$ is the two-mode squeeze operator defined by

$$\hat{S}(\xi_0) = \exp\left[\xi_0^*\hat{a}_1\hat{a}_2 - \xi_0\hat{a}_1^+\hat{a}_2^+\right] \qquad (\xi_0 = s_0\exp(\mathrm{i}\theta_0),\ s_0 > 0) \tag{3.26}$$

for the initial two-mode squeezed coherent state $|\psi(0)\rangle = \hat{D}_1(\alpha_1)\hat{D}_2(\alpha_2)\hat{S}(\xi_0)|0, 0\rangle$, we obtain

$$z_{10} = \alpha_1, \qquad z_{20} = \alpha_2, \qquad \varsigma_0 = r_{120} = 0. \tag{3.27}$$

In the general case, from (3.24), we have

$$\sum_{n_1,n_2} C_{n_1 n_2}|n_1, n_2\rangle = \hat{V}^+(\varsigma_0)\hat{D}_2^+(z_{20})\hat{D}_1^+(z_{10})|\psi(0)\rangle. \tag{3.28}$$

Substitution of expressions (3.21) and (3.28) into the state vector (3.16) results in

$$\begin{aligned}|\psi(t)\rangle = {}&\exp(\mathrm{i}\sigma)\hat{D}_1(z_1)\hat{D}_2(z_2)\hat{V}(\varsigma) \\ &\times\exp\left[-\mathrm{i}\big(\varepsilon_1\hat{a}_1^+\hat{a}_1 + \varepsilon_2\hat{a}_2^+\hat{a}_2\big)\right]\hat{V}^+(\varsigma_0)\hat{D}_2^+(z_{20})\hat{D}_1^+(z_{10})|\psi(0)\rangle.\end{aligned} \tag{3.29}$$

Then, the time-evolution operator for quantum system is

$$\hat{U}(t,0) = \exp(\mathrm{i}\sigma)\hat{D}_1(z_1)\hat{D}_2(z_2)\hat{V}(\varsigma)\exp\left[-\mathrm{i}\big(\varepsilon_1\hat{a}_1^+\hat{a}_1 + \varepsilon_2\hat{a}_2^+\hat{a}_2\big)\right]\hat{V}^+(\varsigma_0)\hat{D}_2^+(z_{20})\hat{D}_1^+(z_{10}). \tag{3.30}$$

In Heisenberg picture, the time evolution of arbitrary operator is given by

$$\hat{A}(t) = \hat{U}^+(t,0)\hat{A}(0)\hat{U}(t,0). \tag{3.31}$$

4. The quantum fluctuations of the quadrature components of the output fields

In this section, we will consider the influence of the two coupled driven harmonic oscillators on the quantum fluctuations of the output fields. It is useful to obtain the time evolution of the annihilation operators. Using equations (3.6), (3.18) and (3.31), as well as following relation (omitting its Hermitian conjugate format)

$$\exp\left[\mathrm{i}\left(\varepsilon_1\hat{a}_1^+\hat{a}_1+\varepsilon_2\hat{a}_2^+\hat{a}_2\right)\right]\hat{a}_j\exp\left[-\mathrm{i}\left(\varepsilon_1\hat{a}_1^+\hat{a}_1+\varepsilon_2\hat{a}_2^+\hat{a}_2\right)\right]=\exp(-\mathrm{i}\varepsilon_j)\hat{a}_j, \tag{4.1}$$

we have

$$\hat{a}_1(t)=\hat{U}^+(t,0)\hat{a}_1\hat{U}(t,0)=h_1+h_{11}\hat{a}_1+h_{12}\hat{a}_2 \tag{4.2a}$$

$$\hat{a}_2(t)=\hat{U}^+(t,0)\hat{a}_2\hat{U}(t,0)=h_2+h_{21}\hat{a}_1+h_{22}\hat{a}_2, \tag{4.2b}$$

where $\hat{a}_j(t)$ is the time-dependent Heisenberg annihilation operation, and

$$h_{11}=\cos r_{12}\cos r_{120}\exp(-\mathrm{i}\varepsilon_1)+\sin r_{12}\sin r_{120}\exp[-\mathrm{i}(\varepsilon_2+\delta_{12}-\delta_{120})] \tag{4.3a}$$

$$h_{12}=\sin r_{12}\cos r_{120}\exp[-\mathrm{i}(\delta_{12}+\varepsilon_2)]-\cos r_{12}\sin r_{120}\exp[-\mathrm{i}(\delta_{120}+\varepsilon_1)] \tag{4.3b}$$

$$h_1=z_1-h_{11}z_{10}-h_{12}z_{20} \tag{4.3c}$$

$$h_{22}=\cos r_{12}\cos r_{120}\exp(-\mathrm{i}\varepsilon_2)+\sin r_{12}\sin r_{120}\exp[\mathrm{i}(\delta_{12}-\delta_{120}-\varepsilon_1)] \tag{4.3d}$$

$$h_{21}=\cos r_{12}\sin r_{120}\exp[\mathrm{i}(\delta_{120}-\varepsilon_2)]-\sin r_{12}\cos r_{120}\exp[\mathrm{i}(\delta_{12}-\varepsilon_1)] \tag{4.3e}$$

$$h_2=z_2-h_{21}z_{10}-h_{22}z_{20}. \tag{4.3f}$$

The creation operators satisfy (4.2) by taking the Hermitian conjugate. We are now interested in the variance of the individual quadrature phase amplitudes [9]

$$\hat{X}_j=\left\{\hat{a}_j^I(t)\exp(\mathrm{i}\phi_{0j})+\left[\hat{a}_j^I(t)\right]^+\exp(-\mathrm{i}\phi_{0j})\right\}/2 \tag{4.4a}$$

$$\hat{Y}_j=\left\{\hat{a}_j^I(t)\exp(\mathrm{i}\phi_{0j})-\left[\hat{a}_j^I(t)\right]^+\exp(-\mathrm{i}\phi_{0j})\right\}/(2\mathrm{i}) \tag{4.4b}$$

and the two-mode quadrature phase amplitudes [9]

$$\hat{X}=\left\{\hat{a}_1^I(t)\exp[\mathrm{i}(\Theta+\varepsilon t)]+\left[\hat{a}_2^I(t)\right]^+\exp[-\mathrm{i}(\Theta-\varepsilon t)]+\mathrm{h.c.}\right\}/2^{3/2} \tag{4.5a}$$

$$\hat{Y}=\left\{\hat{a}_1^I(t)\exp[\mathrm{i}(\Theta+\varepsilon t)]-\left[\hat{a}_2^I(t)\right]^+\exp[-\mathrm{i}(\Theta-\varepsilon t)]-\mathrm{h.c.}\right\}/(2^{3/2}\mathrm{i}), \tag{4.5b}$$

where Θ is the phase of the local oscillator in a homodyne detection scheme. $\hat{a}_j^I(t)=\hat{a}_j(t)\exp(\mathrm{i}\omega_j t)$, $\varepsilon=(\omega_2-\omega_1)/2$. Clearly, we have $[\hat{X}_j,\hat{Y}_j]=\mathrm{i}/2$ and $[\hat{X},\hat{Y}]=\mathrm{i}/2$. It is also not difficult to see that changing ϕ_{0j} in (4.4a) or Θ in (4.5a), both by $\pi/2$, enables one to transform one quadrature into another conjugate . Thus, in the following, we only focus our attention on the quantum fluctuations of $\hat{X}_j$ and $\hat{X}$ quadratures.

If the system starts in the two-mode squeezed number state

$$|\psi(0)\rangle=\hat{S}(\xi_0)|n_1,n_2\rangle \tag{4.6}$$

by using (3.23), (3.25), (4.2)–(4.5), together with the following relational expressions (omitting their Hermitian conjugate formats)

$$\hat{S}^+(\xi_0)\hat{a}_1\hat{S}(\xi_0)=\hat{a}_1\cosh s_0-\hat{a}_2^+\exp(\mathrm{i}\theta_0)\sinh s_0 \tag{4.7a}$$

$$\hat{S}^+(\xi_0)\hat{a}_2\hat{S}(\xi_0)=\hat{a}_2\cosh s_0-\hat{a}_1^+\exp(\mathrm{i}\theta_0)\sinh s_0, \tag{4.7b}$$

the quantum fluctuations (or variances) of $\hat{X}_j$ and $\hat{X}$ in the output fields are given by

$$(\Delta\hat{X}_1)^2 = \{(n_1 - n_2)\cos 2r_{12} + (1 + n_1 + n_2)[\cosh 2s_0 - \cos(\delta_{12} - 2\phi_{01} - \theta_0 + \omega_2 t - \omega_1 t)\sin 2r_{12}\sinh 2s_0]\}/4 \tag{4.8a}$$

$$(\Delta\hat{X}_2)^2 = \{(n_2 - n_1)\cos 2r_{12} + (1 + n_1 + n_2)[\cosh 2s_0 + \cos(\delta_{12} + 2\phi_{02} + \theta_0 + \omega_2 t - \omega_1 t)\sin 2r_{12}\sinh 2s_0]\}/4 \tag{4.8b}$$

$$(\Delta\hat{X})^2 = [(n_2 - n_1)\cos\delta_{12}\sin 2r_{12} + (1 + n_1 + n_2)\{\cosh 2s_0 - [\cos 2r_{12}\cos(2\Theta + \theta_0) + \sin 2r_{12}\sin(2\Theta + \theta_0)\sin\delta_{12}]\sinh 2s_0\}]/4. \tag{4.9}$$

Clearly, $(\Delta\hat{X}_j)^2$ and $(\Delta\hat{X})^2$ are periodic functions of r_{12}, δ_{12}, etc, and dependent on the pump coupling parameters $Q_{12}(t)$ and $\varphi_{12}(t)$, but independent of the driving parameters $G_j(t)$ and $\varphi_j(t)$(see (3.13)). Usually a quantum state is defined squeezed if one of the variances of the quadrature components is less than $1/4$ (which is the quantum fluctuation of the vacuum). For the variances above, when $t = 0$, using (3.25), we get

$$(\Delta\hat{X}_1)^2 = (\cosh 2s_0 + 2n_1\cosh^2 s_0 + 2n_2\sinh^2 s_0)/4 > 1/4 \tag{4.10a}$$

$$(\Delta\hat{X}_2)^2 = (\cosh 2s_0 + 2n_1\sinh^2 s_0 + 2n_2\cosh^2 s_0)/4 > 1/4 \tag{4.10b}$$

$$(\Delta\hat{X})^2 = (1 + n_1 + n_2)[\cosh 2s_0 - \cos\theta_0\sinh 2s_0]/4. \tag{4.11}$$

Obviously, in the input fields, there is no squeezing in the two individual modes but in the two-mode quadrature-phase amplitudes. On the other hand, from (4.8) and (4.9), it is easy to see that the variances vary with interaction time. If n_1 and/or n_2 is large enough and s_0 is small enough, there is no squeezing in all the quadratures; moreover, since r_{12} and δ_{12} are time-dependent functions determined by (3.13), there are always some time points which satisfie $\cos(\delta_{12} - 2\phi_{01} - \theta_0 + \omega_2 t - \omega_1 t)\sin 2r_{12} = 0$, $\cos(\delta_{12} + 2\phi_{02} + \theta_0 + \omega_2 t - \omega_1 t)\sin 2r_{12} = 0$ in (4.8), and $\cos 2r_{12}\cos(2\Theta + \theta_0) + \sin 2r_{12}\sin(2\Theta + \theta_0)\sin\delta_{12} = 0$ in (4.9). This allows all the variances being larger than $1/4$ for arbitrary n_1, n_2 and s_0. However, at more time points, the expressions above are not equal to zero; in general, we can always find some time region in which the expressions above equal and approximately equal ± 1. Thus, as long as s_0 is large enough, regardless of what n_1 and n_2 equal, the squeezing occurs periodically in all the quadratures and it can be interchanged between each pair of the conjugate components. So we may conclude that, in some particular conditions, the time evolution of the two coupled driven harmonic oscillators can not only preserve the initial two-mode squeezing, but will also produce squeezing even though there is no initial squeezing in the two individual modes. Now we again consider the quantum fluctuations of the output fields in a frequency converter in the case of perfect energy matching. By inserting (3.14) into (4.8) and (4.9), we get

$$(\Delta\hat{X}_1)^2 = \{(n_1 - n_2)\cos(2Q_{12}t) + (1 + n_1 + n_2) \times [\cosh 2s_0 - \sin(2\phi_{01} + \theta_0)\sin(2Q_{12}t)\sinh 2s_0]\}/4 \tag{4.12a}$$

$$(\Delta\hat{X}_2)^2 = \{(n_2 - n_1)\cos(2Q_{12}t) + (1 + n_1 + n_2) \times [\cosh 2s_0 - \sin(2\phi_{02} + \theta_0)\sin(2Q_{12}t)\sinh 2s_0]\}/4 \tag{4.12b}$$

$$(\Delta\hat{X})^2 = [(n_2 - n_1)\sin(2Q_{12}t)\sin\omega t + (1 + n_1 + n_2)\{\cosh 2s_0 + [\cos(2\Theta + \theta_0)\cos(2Q_{12}t) + \sin(2\Theta + \theta_0)\sin(2Q_{12}t)\cos\omega t]\sinh 2s_0\}]/4. \tag{4.13}$$

In this case, $(\Delta\hat{X}_j)^2$, which varies sinusoidally with time, is independent of the frequency of the pump field ω. For some special parameter values, the squeezing can be generated periodically with period π/Q_{12}. For example, when $2\phi_{0j} + \theta_0 = \pi/2$, $s_0 > \frac{1}{2}\ln(1+n_1+n_2)$, and $t = \pi/(4Q_{12}), 5\pi/(4Q_{12}), \ldots$, we have $(\Delta\hat{X}_1)^2 = (\Delta\hat{X}_2)^2 = (1+n_1+n_2)\exp(-2s_0)/4 < 1/4$. However, $(\Delta\hat{X})^2$ exhibits a rapidly oscillatory behaviour at a high frequency ω. If $2\Theta + \theta_0$ equals 0 or π; and $n_1 = n_2$ or $|n_1 - n_2|$ is small enough, or $n_1 = n_2 = 0$ (which corresponds to a initial squeezed vacuum state), we come to the same conclusion as above, namely, the squeezing can be generated periodically with time at the frequency of $2Q_{12}$.

If the system starts in the two-mode squeezed coherent state

$$|\psi(0)\rangle = \hat{D}(\alpha_1)\hat{D}(\alpha_2)\hat{S}(\xi_0)|0,0\rangle \tag{4.14}$$

using (3.18c), (3.27), (4.2)–(4.5), (4.7) and (3.23), we obtain the variances of $\hat{X}_j$ and $\hat{X}$ as follows:

$$(\Delta\hat{X}_1)^2 = [\cosh 2s_0 - \cos(\delta_{12} + \omega_2 t - \omega_1 t - 2\phi_{01} - \theta_0)\sin 2r_{12}\sinh 2s_0]/4 \tag{4.15a}$$

$$(\Delta\hat{X}_2)^2 = [\cosh 2s_0 + \cos(\delta_{12} + \omega_2 t - \omega_1 t + 2\phi_{02} + \theta_0)\sin 2r_{12}\sinh 2s_0]/4 \tag{4.15b}$$

$$\langle\Delta\hat{X}^2\rangle = \{\cosh 2s_0 - [\cos 2r_{12}\cos(2\Theta + \theta_0) + \sin 2r_{12}\sin(2\Theta + \theta_0)\sin\delta_{12}]\sinh 2s_0\}/4. \tag{4.16}$$

The most striking feature of (4.15) and (4.16) is that they are independent of the initial values α_1 and α_2, and take the special forms of (4.8) and (4.9) respectively in the case of $n_1 = n_2 = 0$. This indicates that the quantum fluctuations of the output fields for the initial two-mode squeezed coherent state equal to those for the initial two-mode squeezed vacuum state. For the frequency converter, (4.15) and (4.16) reduce to

$$(\Delta\hat{X}_1)^2 = [\cosh 2s_0 - \sin(2\phi_{01} + \theta_0)\sin(2Q_{12}t)\sinh 2s_0]/4 \tag{4.17a}$$

$$(\Delta\hat{X}_2)^2 = [\cosh 2s_0 - \sin(2\phi_{02} + \theta_0)\sin(2Q_{12}t)\sinh 2s_0]/4 \tag{4.17b}$$

$$(\Delta\hat{X})^2 = \{\cosh 2s_0 - [\cos(2\Theta + \theta_0)\cos(2Q_{12}t) + \sin(2\Theta + \theta_0) \times \sin(2Q_{12}t)\cos\omega t]\sinh 2s_0\}/4. \tag{4.18}$$

Evidently, when $2\phi_{0j} + \theta_0$ equal $\pi/2$ or $3\pi/2$ in (4.17) and $2\Theta + \theta_0$ equal 0 or π in (4.18), $(\Delta\hat{X}_1)^2$, $(\Delta\hat{X}_2)^2$ and $\langle\Delta\hat{X}^2\rangle$ all exhibit periodical squeezing behaviour, the period and the minimum are π/Q_{12} and $\exp(-2s_0)/4$ respectively.

In (4.6), if $s_0 = 0$, we have $|\psi(0)\rangle = \hat{S}(\xi_0)|n_1, n_2\rangle = |n_1, n_2\rangle$. Thus, for the initial two-mode number state, (4.8) and (4.9) reduce to

$$(\Delta\hat{X}_1)^2 = (1 + 2n_1\cos^2 r_{12} + 2n_2\sin^2 r_{12})/4 \tag{4.19a}$$

$$(\Delta\hat{X}_2)^2 = (1 + 2n_2\cos^2 r_{12} + 2n_1\sin^2 r_{12})/4 \tag{4.19b}$$

$$(\Delta\hat{X})^2 = [1 + n_1(1 - \cos\delta_{12}\sin 2r_{12}) + n_2(1 + \cos\delta_{12}\sin 2r_{12})]/4. \tag{4.20}$$

Clearly, for arbitrary n_1, n_2, r_{12} and δ_{12}, the variances of all the quadrature phase amplitudes above are all larger than 1/4. Therefore, there is no squeezing in this case.

In (4.14), if $s_0 = 0$, we obtain $|\psi(0)\rangle = \hat{D}_1(\alpha_1)\hat{D}_2(\alpha_2)\hat{S}(\xi_0)|0,0\rangle = |\alpha_1, \alpha_2\rangle$. Thus, for the initial two-mode coherent state, (4.15) and (4.16) reduce to $(\Delta\hat{X}_j)^2 = (\Delta\hat{X})^2 = 1/4$, so do $(\Delta\hat{Y}_j)^2 = (\Delta\hat{Y})^2$. Accordingly, there is no squeezing in all the quadrature-phase amplitudes of the output fields either; and the system stays in a minimum-uncertainty state. The state of the system therefore remains coherent.

5. Summary and conclusions

Using Lewis–Riesenfeld invariant theory and selecting a suitable time-dependent unitary transformation to the Hermitian operator $\hat{K}_0 = \sum_{j=1}^{2} C_j \hat{a}_j^+ \hat{a}_j$, we have obtained the state vector and the time evolution operator for the two forced quantum oscillators with mixing of two modes, and investigated the quantum fluctuation of the quadrature-phase amplitudes in the output fields for various initial states. The following results are obtained.

(1) When the system starts in the two-mode squeezed number or squeezed coherent state, in some particular conditions, the time evolution of the oscillators not only preserves the initial two-mode squeezing, but also periodically produces squeezing in the individual modes. Furthermore, the squeezing can be interchanged between each pair of the conjugate components as the interaction time increases.

(2) The variances of the output fields for the initial squeezed coherent state $|\psi(0)\rangle = \hat{D}(\alpha_1)\hat{D}(\alpha_2)\hat{S}(\xi_0)|0, 0\rangle$ are independent of the initial values α_1 and α_2, and equal the variances for the initial two-mode squeezed vacuum state.

(3) For the case of perfect energy matching in a frequency converter, whether the system starts in a two-mode squeezed number state or a squeezed coherent state, for some specific values of ϕ_{0j}, Θ, etc, all the variances may be independent of the frequency of the pump field ω, vary sinusoidally with the interaction time, and exhibit a rather long periodical (with a period π/Q_{12}) squeezing behaviour since Q_{12} is generally small.

(4) When the system starts in the two-mode number state or in the two-mode coherent state, the quantum fluctuation of all the quadrature phases in the output fields is not less than $1/4$. Hence, there is no squeezing in the individual and mutual quadrature-phase amplitudes of the output two-mode fields.

(5) Whichever of the four states above the system starts in, the quantum fluctuations of all the quadrature phases in the output fields are all independent of the driving parameters $G_j(\mathrm{t})$ and $\varphi_j(\mathrm{t})$. In particular, for the initial two-mode coherent state, the variances of the output fields preserve their initial values $1/4$, and are also independent of other parameters in the Hamiltonian $\hat{H}$.

It should be noted that we have only derived a closed solution of the Schrödinger equation and obtained a special solution of a frequency converter in the case of perfect energy matching in this study. In a forthcoming paper, we will consider an explicit analytical solution for a generalized frequency conversion process.

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